# Performance Guarantees for Distributed MIMO Radar based on Sparse Sensing

**Bo Li and Athina P. Petropulu**

**Electrical & Computer Engineering, Rutgers, The State University of New Jersey**

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## Overview

- We consider sparse sensing-based distributed MIMO radars, which exploit the sparsity of the targets in the space to achieve good target estimation performance of MIMO radars but with fewer measurements.
- In the model of sparse sensing-based distributed MIMO radars, the sensing matrix is block-diagonal and the sparse vector to be recovered consists of equal-length sub-vectors that have the same sparsity profile.
- This paper develops the theoretical requirements and performance guarantees for the application of block sparse recovery technique in this context.
- The results confirm that exploiting the block sparsity of the target vector can reduce the number of measurements needed for target estimation, or can result in improved target estimation for the same number of measurements.

## System Model

The location-speed space is discretized by $\Theta$ with $N$ grid points. For the $(ij)$-th TX/RX pair, the signal vector at $j$-th RX from $P$ pulses due to the transmission of $i$-th TX

$$\mathbf{z}_{ij} = \mathbf{\Psi}_{ij}\,\mathbf{s}_{ij} + \mathbf{n}_{ij}, \quad \forall i \in \mathbb{N}^+_{M_t}, j \in \mathbb{N}^+_{M_r}$$

where $\mathbf{s}_{ij} = [s_{ij}{}^1, \ldots, s_{ij}{}^N]^T$ with $s^n_{ij}$ being non-zero only if there is a target at the $n$-th grid point; and

$$\mathbf{\Psi}_{ij} = \begin{bmatrix} \mathbf{x}_{i,\tau^1_{ij}} e^{j2\pi f^1_{ij}T} & \cdots & \mathbf{x}_{i,\tau^N_{ij}} e^{j2\pi f^N_{ij}T} \\ \vdots & \ddots & \vdots \\ \mathbf{x}_{i,\tau^1_{ij}} e^{j2\pi f^1_{ij}PT} & \cdots & \mathbf{x}_{i,\tau^N_{ij}} e^{j2\pi f^N_{ij}PT} \end{bmatrix}_{(LP\times N)}$$

where $\mathbf{x}_{i,\tau^n_{ij}}$ is a vector contains $L$ samples of the $i$-th waveform shifted by $\tau^n_{ij}$. $n_{ij}$ represents noise. $\tau^n_{ij}$ and $f^n_{ij}$ denote the delay time and Doppler frequency. The transmitted waveforms are Gaussian signals with variance $\sigma^2_0$.

Stacking the received samples into a vector $\mathbf{z}$, we get

$$\mathbf{z} = \left[\mathbf{z}^T_{11}, \ldots, \mathbf{z}^T_{M_tM_r}\right]^T = \mathbf{\Psi}\mathbf{s}+\mathbf{n} \qquad (1)$$

where $\mathbf{s} = \left[\mathbf{s}^T_{11}, \ldots, \mathbf{s}^T_{M_tM_r}\right]^T$, $\mathbf{n} = \left[\mathbf{n}^T_{11}, \ldots, \mathbf{n}^T_{M_tM_r}\right]^T$ and $\mathbf{\Psi} = \text{diag}(\mathbf{\Psi}_{11}, \ldots, \mathbf{\Psi}_{M_tM_r})$.

The vector $\mathbf{s}$ is a concatenation of $M_tM_r$ sub-vectors that share the same sparsity profile, and have exactly $K$ nonzero entries each. $\mathbf{s}$ lies in $\mathcal{A}^K_0$ defined by

$$\mathcal{A}^K_0 \equiv \{\mathbf{y} \in \mathbb{C}^{NM_tM_r} | \text{supp}(\mathbf{y}_1) = \cdots = \text{supp}(\mathbf{y}_{M_tM_r}), |\text{supp}(\mathbf{y}_j)| \le K\}$$

## *Sparse Signal Recovery*

- By directly applying the L-OPT in [1], we have

$$\min_{\mathbf{s}} \sum_{n=1}^{N} \|\mathbf{s}[\mathcal{I}_n]\|_2 \quad s.t.\ \|\mathbf{z} - \mathcal{P}_M(\mathbf{\Psi})\mathcal{P}_v(\mathbf{s})\|_2 \le \epsilon \qquad (2)$$

where

- $\mathcal{P}_v(\mathbf{s}) = [\mathbf{s}[\mathcal{I}_1]; \ldots; \mathbf{s}[\mathcal{I}_N]], \mathcal{A}^K_{blk} \equiv \{\mathcal{P}_v(\mathbf{s}) | \mathbf{s} \in \mathcal{A}^K_0\}$;
- $\mathcal{P}_M(\mathbf{\Psi})$: permutation of columns of $\mathbf{\Psi}$.

**Definition 1**: Matrix $\mathbf{\Psi}$ satisfies the RIP over $\mathcal{A}$ with $\delta_K$, or equivalently the $\mathcal{A}$-$RIP(K, \delta_K)$, if for every $\mathbf{x} \in \mathcal{A}$ it holds that $(1 - \delta_K\|\mathbf{x}\|^2_2 \le \|\mathbf{\Psi}\mathbf{x}\|^2_2 \le (1 + \delta_K\|\mathbf{x}\|^2_2))$.

**Result 1:** Consider $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi}/(\sqrt{LP}\sigma_0)$. If $\mathcal{P}_M(\widetilde{\mathbf{\Psi}})$ satisfies the $\mathcal{A}_{blk}$-RIP$(2K, \delta_{2K})$ with $\delta_{2K} \le \sqrt{2} - 1$, then the L-OPT method in (2) can recover $\mathbf{s}$ with

$$\|\hat{\mathbf{s}} - \mathbf{s}\|_2 \le 4\sqrt{1 + \delta_{2K}}/\left(1 - \left(1 + \sqrt{2}\right)\delta_{2K}\right)\epsilon.$$

- To prove the $\mathcal{A}_{blk}$-RIP of $\widetilde{\mathbf{\Psi}}$, we utilize the fact that $\mathcal{A}_1$-RIP of $\widetilde{\mathbf{\Psi}} \rightarrow \mathcal{A}_0$-RIP of $\widetilde{\mathbf{\Psi}} \leftrightarrow \mathcal{A}_{blk}$-RIP of $\mathcal{P}_M(\widetilde{\mathbf{\Psi}})$ where $\mathcal{A}^K_0 \subset \mathcal{A}^K_1$ and $\mathcal{A}^K_1$ is defined as

$$\mathcal{A}^K_1 \equiv \{\mathbf{y} \in \mathbb{C}^{NM_tM_r} |\ |\text{supp}(\mathbf{y}_1)| = \cdots = |\text{supp}(\mathbf{y}_{M_tM_r})| \le K\}.$$

## Observations on the Gram of $\widetilde{\mathbf{\Psi}}$

The Gram of $\widetilde{\mathbf{\Psi}}$ is denoted by $\mathbf{G} = \text{diag}(\mathbf{G}_{11}, \ldots, \mathbf{G}_{M_tM_r})$ where $\mathbf{G}_{ij} = \widetilde{\mathbf{\Psi}}^H_{ij}\widetilde{\mathbf{\Psi}}_{ij}$. The $(n, m)$-th entry of $\mathbf{G}_{ij}$ equals

$$|\mathbf{G}_{ij}(n, m)| = \frac{\mathbf{x}^T_{i,\tau^n_{ij}}\mathbf{x}_{i,\tau^m_{ij}}}{LP\sigma^2_0} \left|\frac{\sin(\pi(f^m_{ij} - f^n_{ij})TP)}{\sin(\pi(f^m_{ij} - f^n_{ij})T)}\right| e^{j2\pi(f^m_{ij} - f^n_{ij})TP}$$

To bound the entries of $\mathbf{G}_{ij}$, we have three cases

- **Case (i)**: $n = m$, i.e., diagonal entries $\mathbf{G}_{ij}(n)$,

$$\Pr(|\mathbf{G}_{ij}(n) - 1| > t) \le 2\exp(-Lt^2/16)$$

- **Case (ii)**: $\tau^n_{ij} \ne \tau^m_{ij}$, i.e., off-diagonal entries,

$$\Pr(|\mathbf{G}_{ij}(n, m) > t|) \le 4\exp\left(-\frac{(L-1)t^2}{8 + 4t}\right)$$

- **Case (iii)**: $\tau^n_{ij} = \tau^m_{ij}, f^n_{ij} \ne f^m_{ij}$, we have

$$|\mathbf{G}_{ij}(n, m)| = \frac{\mathbf{x}^T_{i,\tau^n_{ij}}\mathbf{x}_{i,\tau^n_{ij}}}{LP\sigma^2_0} \left|\frac{\sin(\pi(f^m_{ij} - f^n_{ij})TP)}{\sin(\pi(f^m_{ij} - f^n_{ij})T)}\right|$$

Denoting the second multiplier as $\phi^{mn}_{ij}$, $|\mathbf{G}_{ij}(n, m)|$ can be viewed as a squared norm of a Gaussian vector. Applying Lemma 5 in [2], we have

$$\Pr(|\mathbf{G}_{ij}(n, m) > t|) \le \exp\left(-\frac{L}{16}\left(t/\gamma_{ij} - 1\right)^2\right)$$

where $\gamma_{ij} \equiv \sup_{m,n \in S_2} |\phi^{mn}_{ij}/P|$,

$$S_2 \equiv \{(m, n) | m, n \le N, \tau^n_{ij} = \tau^m_{ij}, f^n_{ij} \ne f^m_{ij}\}.$$

## Measurement Matrix Satisfying $\mathcal{A}_1$-RIP

**Theorem 1:** For any $\delta_K \in (0.1)$, there exist $c_1$ and $c_2$ such that $\widetilde{\mathbf{\Psi}}$ satisfies $\mathcal{A}_1$-RIP$(K, \delta_K)$ with probability exceeding $1 - \exp(-c_1(L-1)/K^2)$ whenever

$$L \ge c_2K^2\log(NM_tM_r) + 1, \qquad (3)$$

$$\gamma_{ij} \le \frac{\delta_K}{2K + \delta_K}, \forall i \in \mathbb{N}^+_{M_t}, j \in \mathbb{N}^+_{M_r} \qquad (4)$$

*Sketch of proof*: Under condition (4), the bounds on the off-diagonal entries from case (ii) and (iii) are unified by

$$\Pr(|\mathbf{G}_{ij}(n, m) > t|) \le 4\exp\left(-\frac{(L-1)t^2}{16}\right).$$

Applying the Gergosin's Disc Theorem proves the claims.

**Remark**: Exploiting the structures in both $\mathbf{\Psi}$ and $\mathbf{s}$ allows for reduction of the number of samples, $L$, needed for target estimation. From [2], a full Toeplitz matrix satisfies the RIP if $L$ is on the order of $\mathcal{O}(K^2M_tM_r\log(NM_tM_r))$, which is $M_tM_r$ times larger than the bound in (3).

## Numerical Results

We consider a MIMO radar system with $M_t = 2$ TX and $M_r = 2$ RX antennas, distributed uniformly on a circle of radius of $6km$ and $3km$, respectively. The probing space is discretized on a $20 \times 4$ grid, with grid spacing equal to $10\ m$. The velocity space is $V_x \in [100, 130]m/s, V_y = 100m/s$ and is uniformly discretized on a $4 \times 1$ grid. AWGN with variance $\sigma^2_n$ is considered. and the SNR is defined as $10\log_{10}(\sigma^2_0/\sigma^2_n)$.

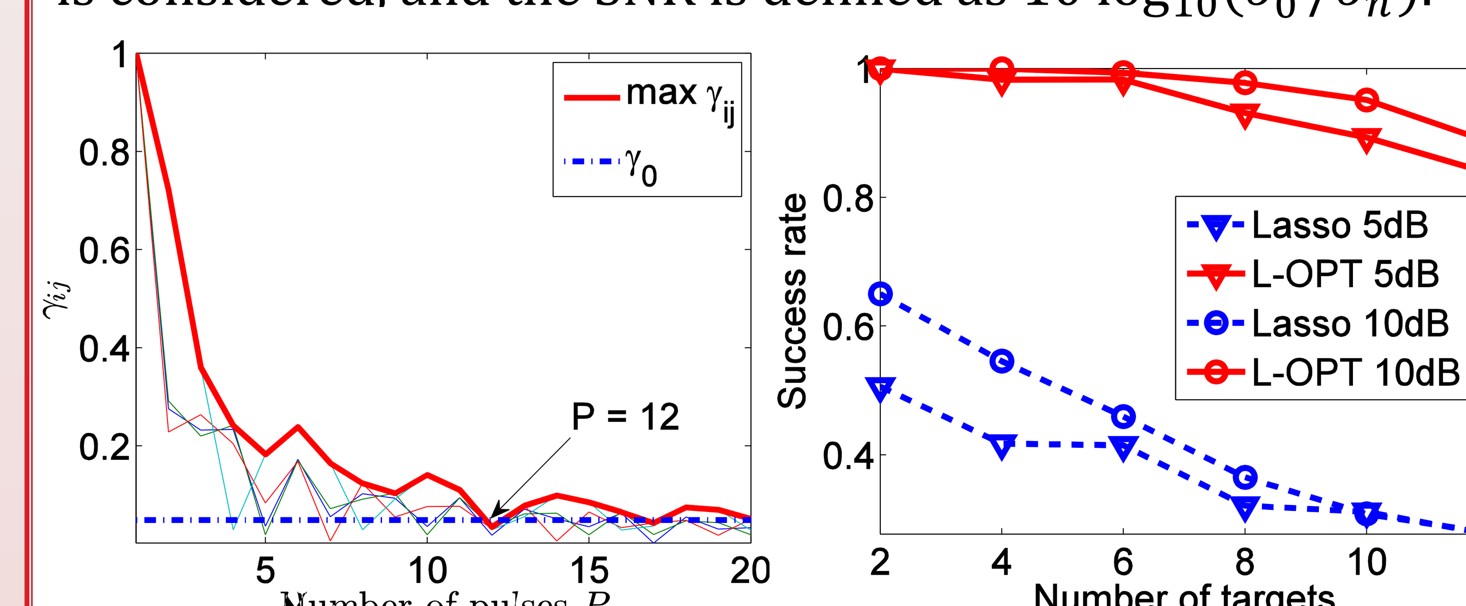


Fig. 1. **Left**: Results on the choice of the number of pulses, $P$; **Right:** success recovery rate for different number of targets, $K$, with $L = 6, P = 3$.

## *References*

[1] Y. C. Eldar and M. Mishali, "Robust recovery of signals from a structured union of subspaces." IEEE Trans. Inf. Theory, vol. 55, no. 11, pp. 5302-5316, Nov. 2009.

[2] J. Haupt, W. U. Bajwa, G. Raz, and R. Nowak, "Toeplitz compressed sensing matrices with applications to sparse channel estimation," IEEE Trans. Inform. Theory, vol. 56, no. 11, pp. 5862–5875, 2010.