# STRUCTURED SAMPLING OF STRUCTURED SIGNALS

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## Overview

- We consider the structured sampling of structured signals, more specifically, using block diagonal (BD) measurement matrices to sense signals with uniform partitions that share the same sparsity profile. This model arises in distributed compressive sensing systems.
- We are interested in the efficient recovery of the sparse signal and the corresponding performance as determined by the restricted isometry property (RIP) of the measurement matrix.
- We characterize the RIP of the random BD matrix with respect to signals with the aforementioned structure.
- We study the multiple measurement vector (MMV) problem as a special case of the general problem considered here.

## System Model and Background

System model

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_J \end{bmatrix}}_{\mathbf{y}\in\mathbb{R}^{\tilde{M}}} = \underbrace{\begin{bmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_J \end{bmatrix}}_{\Psi\in\mathbb{R}^{\tilde{M}\times\tilde{N}}} \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_J \end{bmatrix}}_{\mathbf{x}\in\mathbb{R}^{\tilde{N}}} \qquad (1)$$

where

$$\mathbf{y}_j \in \mathbb{R}^M, \Phi_j \in \mathbb{R}^{M\times N}, \mathbf{x}_j \in \mathbb{R}^N, \tilde{M} = JM, \tilde{N} = JN$$

- The sparse signal vector $\mathbf{x} \triangleq [\mathbf{x}_1{}^T, \dots, \mathbf{x}_J{}^T]^T$ lies in
  $$\mathcal{A}_0^K \triangleq \{\mathbf{x} \in \mathbb{R}^{\tilde{N}} \mid \text{supp}(\mathbf{x}_1) = \dots = \text{supp}(\mathbf{x}_J), |\text{supp}(\mathbf{x}_j)| \leq K\}$$
- This model arises in distributed compressive sensing systems for time-variant sparse channel estimation, hyper-spectral imaging and sensor networks.

Restricted isometry property of a general matrix $\mathbf{\Psi}$ with respect to signals in $\mathcal{A}_0^K$ ($\mathcal{A}_0$-*RIP*):

Matrix $\mathbf{\Psi} \in \mathbb{R}^{\tilde{M}\times\tilde{N}}$ satisfies the RIP over $\mathcal{A}_0^K$ with constant $\delta_K$, or equivalently the $\mathcal{A}_0$-*RIP*$(K, \delta_K)$, if for every $\mathbf{x} \in \mathcal{A}_0^K$ it holds that

$$(1 - \delta_K)\|\mathbf{x}\|_2^2 \leq \|\Psi\mathbf{x}\|_2^2 \leq (1 + \delta_K)\|\mathbf{x}\|_2^2$$

- The above definition can be applied for general union of subspaces including $\mathcal{A}_0^K$.

## Random BD Matrices Satisfying $\mathcal{A}$-RIP

- **Lemma 1**: concentration of random BD matrices

Let $\mathbf{\Psi}$ be a BD matrix as defined in (1) with entries of distribution $\mathcal{N}(0, 1/M)$. Then for any fixed $\mathbf{x} \in \mathbb{R}^{\tilde{N}}$

$$\mathbf{P}(|\,\|\Psi\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2\,| \geq \epsilon\|\mathbf{x}\|_2^2) \leq 2e^{-cM\epsilon^2}$$

where $c$>0 is a constant, $\epsilon$>0 is a small value.

- **Theorem 1:** $\mathcal{A}_0$-RIP for random BD matrices

The block diagonal matrix $\mathbf{\Psi} \in \mathbb{R}^{\tilde{M}\times\tilde{N}}$ satisfies the $\mathcal{A}_0$-*RIP*$(K, \delta_K)$ for some $\delta_K \in (0,1)$, $t$>0 and

$$M \geq \frac{c_1}{\delta_K^2}\left(K\log\frac{N}{K} + JK\log(e(1 + \frac{12}{\delta_K})) + \log 2 + t\right) \qquad (2)$$

with probability at least $1 - e^{-t}$, where $c_1 < 9/c$.

- **Corollary 1**: standard RIP for random BD matrices

The BD matrix $\mathbf{\Psi} \in \mathbb{R}^{\tilde{M}\times\tilde{N}}$ satisfies the RIP$(K, \delta_K)$ for some $\delta_K \in (0,1)$, t>0 with probability at least 1-$e^{-t}$ if

$$M \geq \frac{c_1}{\delta_K^2}\left(K\log\frac{\tilde{N}}{K} + K\log(e(1 + \frac{12}{\delta_K})) + \log 2 + t\right)$$

## *Sparse Signal Recovery*

### $l_0$–norm solution

$$(P_0): \min_{\mathbf{x}} \sum_{n=1}^{N} I\left(\|\mathbf{x}[\mathcal{I}_n]\|_2\right) \quad \text{s.t.} \quad \mathbf{y} = \Psi\mathbf{x}$$

where

- $\mathcal{I}_n$ contains the $n$-th entries of all $\mathbf{x}_j, j = 1, \dots, J$;
- $I(a) = \begin{cases} 1, & a \neq 0 \\ 0, & a = 0 \end{cases}$.

Uniqueness of $l_0$–norm solution is guaranteed as follows: $\mathbf{x} \in \mathcal{A}_0^K$ can be uniquely recovered by solving $(P_0)$, if $\mathbf{\Psi}$ satisfies the $\mathcal{A}_0$-*RIP*$(2K, \delta_{2K})$ with $\delta_{2K}$<1.

### $l_1$–norm solution

- By directly applying the L-OPT in [1], we have

$$(P_1): \min_{\mathbf{x}} \sum_{n=1}^{N} \|\mathbf{x}[\mathcal{I}_n]\|_2 \quad \text{s.t.} \quad \mathbf{y} = \mathcal{P}_M(\Psi)\mathcal{P}_v(\mathbf{x})$$

where

- $\mathcal{P}_v(\mathbf{x}) = [\mathbf{x}[\mathcal{I}_1]; \dots; \mathbf{x}[\mathcal{I}_N]]$, $\mathcal{A}_{blk}^K \triangleq \{\mathcal{P}_v(\mathbf{x}) \mid \mathbf{x} \in \mathcal{A}_K\}$;
- $\mathcal{P}_M(\mathbf{\Psi})$: permutation of columns of $\mathbf{\Psi}$.

**Proposition 1**: $\mathcal{P}_M(\mathbf{\Psi})$ satisfies the $\mathcal{A}_{blk}$-RIP$(K, \delta_K)$ ***if and only if*** $\mathbf{\Psi}$ satisfies the $\mathcal{A}_0$-*RIP*$(K, \delta_K)$.

If the condition in (2) holds with $\delta_{2K}$<$\sqrt{2}$-1, i.e., the BD matrix $\mathbf{\Psi}$ satisfies the $\mathcal{A}_0$-*RIP*$(2K, \delta_{2K})$, $(P_1)$ can uniquely recover the sparse vector $\mathbf{x}$ in $\mathcal{A}_0^K$ [1].

## Proposed Application to the MMV Model

The multiple measurement vector (MMV) model

$$\mathbf{Y} = \mathbf{MX}, \ \mathbf{M} \in \mathbb{R}^{M\times N}, \mathbf{X} \in \mathbb{R}^{N\times J} \qquad (3)$$

- $\mathbf{Y}$ has $J$ columns containing measurements, $\mathbf{X}$ has $K$ nonzero rows;
- SMV: special case of MMV for $J$=1.

We reformulate the MMV model as follows

$$(P_2): \text{vec}(\mathbf{Y}) = \text{diag}(\mathbf{M}, \dots, \mathbf{M})\text{vec}(\mathbf{X}), \quad \text{vec}(\mathbf{X}) \in \mathcal{A}_0^K$$

If the condition in (2) holds with $\delta_{2K}$<$\sqrt{2}$-1, i.e., the BD matrix diag($\mathbf{M}$, … , $\mathbf{M}$) satisfies the $\mathcal{A}_0$-RIP$(2K, \delta_{2K})$, applying $(P_1)$ will uniquely recover $\mathbf{X}$ with probability at least 1-$e^{-t}$.

- For MMV $M$ is $\mathcal{O}\left(K\log\frac{N}{K} + JK\right)$; for SMV $M$ is $\mathcal{O}(K\log\frac{N}{K})$.

The above result is the first strict RIP-based worst case analysis for the MMV model .

Remark: The MMV results in [1, Section VI.A] is not applicable for random $\mathbf{M}$.

## Comparison with Different CS Models

Let us consider 4 different models that consider structures in $\mathbf{\Psi} \in \mathbb{R}^{\tilde{M}\times\tilde{N}}$ and/or $\mathbf{x} \in \mathbb{R}^{\tilde{M}\times\tilde{N}}$ with $JK$ nonzero entries:

- *Standard CS* [2] with a dense random matrix $\mathbf{\Psi}$ and an arbitrarily $JK$-sparse $\mathbf{x}$ requires $\tilde{M} = \mathcal{O}(JK\log\frac{N}{K})$
- *Model based CS* [1,3] with a dense random matrix $\mathbf{\Psi}$ and $\mathbf{x} \in \mathcal{A}_{blk}^K$ requires $\tilde{M} = \mathcal{O}(K\log\frac{N}{K} + JK)$
- *Model in Corollary 1* with BD random matrix $\mathbf{\Psi}$ and arbitrarily $JK$-sparse $\mathbf{x}$ requires $\tilde{M} = \mathcal{O}(J^2K\log\frac{N}{K})$
- *Distributed CS* in this paper with BD random matrix $\mathbf{\Psi}$ and $\mathbf{x} \in \mathcal{A}_0^K$ requires $\tilde{M} = \mathcal{O}(JK\log\frac{N}{K} + J^2K)$.

## *References*

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